Journal of Applied Mechanics and Technical Physics, Vol. 48, No. 2, pp. 192–199, 2007

IDEAL INCOMPRESSIBLE FLOW AROUND A WEDGE TIP

S. K. Betyaev

UDC 532.526.5

A planar analog of conical flows is considered: an inviscid incompressible fluid flow around a wedge tip. A class of conical flows is found where vorticity is transported along streamlines by the potential component of velocity. Problems of a wave "locked" in the corner region and of a flow accelerating along the rib of a dihedral angle are considered. By analogy with an axisymmetric quasi-conical flow, a planar quasi-conical flow of the fluid is determined, namely, the flow inside and outside the region bounded by tangent curves described by a power law. Conditions are found where vorticity and swirl produce a significant effect. An approximate solution of the problem of the fluid flow inside a "zero" angle is obtained.

Key words: *ideal fluid, planar conical and quasi-conical flows, vorticity, unsteadiness.*

1. Formulation of the Problem. In the case of a conical flow, the problem dimension decreases by one. Such a decrease in the problem dimension is known to significantly simplify the problem and allow a more detailed investigation of the problem [1, 2].

The following types of conical flows exist.

1. Steady conical flow of an ideal fluid. In this case, the velocity is independent of the radial coordinate (it is constant on the rays $\theta = \text{const}$ and $\varphi = \text{const}$ of a spherical coordinate system). The solution is exact and is presented as

$$\boldsymbol{u}(r,\theta,\varphi) = \boldsymbol{U}(\theta,\varphi), \qquad p(r,\theta,\varphi) = P(\theta,\varphi).$$

For an axisymmetric flow, we have $\partial \boldsymbol{U}/\partial\varphi = \partial P/\partial\varphi = 0$.

2. Unsteady conical flow of an ideal fluid. In this case, the exact solution of the Euler equations has the form

$$\boldsymbol{u}(r,\theta,\varphi,t) = r\boldsymbol{U}(\theta,\varphi,t), \qquad p(r,\theta,\varphi,t) = r^2 P(\theta,\varphi,t).$$

3. Steady conical flow of a viscous fluid described by the exact solution of the Navier–Stokes equations:

$$\boldsymbol{u}(r,\theta,\varphi) = \frac{1}{r}\,\boldsymbol{U}(\theta,\varphi), \qquad p(r,\theta,\varphi) = \frac{1}{r^2}\,P(\theta,\varphi).$$

Examples of such flows are the Hamel solution, the Long vortex, and the Landau jet [3].

4. Generalized conical flows of an ideal fluid [4]. In this case, the velocity is presented as

$$\boldsymbol{u}(r,\theta,\varphi) = r^n\boldsymbol{U}(\theta,\varphi), \qquad p(r,\theta,\varphi) = r^{2n}P(\theta,\varphi),$$

where n is the conicity index.

5. Quasi-conical flows [5, 6], where

$$\boldsymbol{u}(x,y,z,t) = r^n\boldsymbol{U}\Big(\frac{y}{x^m},\frac{z}{x^m}\Big) + o(r^n), \qquad p(x,y,z,t) = r^{2n}P\Big(\frac{y}{x^m},\frac{z}{x^m}\Big) + o(r^{2n})$$

(m is the quasi-conicity index).

Joukowski Central Aerohydrodynamic Institute, Zhukovskii 140181; betyaevs@gmail.com. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 48, No. 2, pp. 57–65, March–April, 2007. Original article submitted October 25, 2005; revision submitted February 26, 2006.

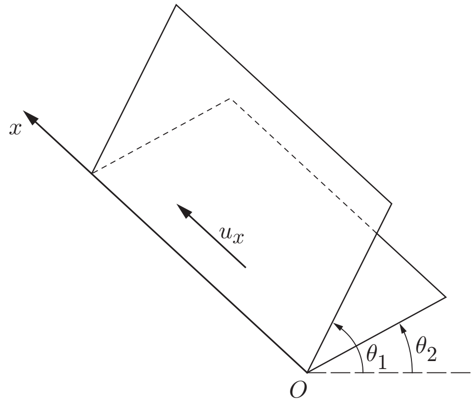

Fig. 1. Coordinate system.

Not being an exact solution, the quasi-conical solution is a coordinate expansion in the neighborhood of the point $y = z = x = 0$.

The singularity is concentrated at the cone apex in the case of a conical flow and on the line $r = 0$, which is the apex of a dihedral angle, in the case of a planar conical flow; in the latter case, the solution is constructed as a coordinate series with respect to the radial distance from the line r.

The solution of the problem of an axisymmetric flow around the cone apex, like the solution of the problem of the flow around the wedge apex, is unknown within the framework of the Navier–Stokes model [7]. The local solution of the Euler equations yields the "outer" limit of this yet unsolved problem. The local solution of the Navier–Stokes equation is a complicated multi-layer structure, while the local solution of the Euler equation is a single-layer coordinate expansion.

Nonuniqueness of solutions is more inherent to the local theory of an ideal fluid rather than to the global theory. The reason is that the conditions of matching with the external expansion are not satisfied in the local (internal) theory. In the local theory, a cone or a wedge are extended to infinity, whereas real bodies have finite sizes. For this reason, apparently, the local problem has no solution: either the solution becomes singular at a certain fixed value of the parameter (Sternberg–Koiter paradox [8]) or the solution does not exist at all in a certain range of the values of the governing parameter.

The objective of the present work is to study new formulations of the problems of the theory of planar conical and quasi-conical flows: transport of vorticity by a potential flow, effect of unsteadiness, linear wave, and flow inside and outside a "zero" angle.

The Euler equations for an incompressible fluid of unit density in a cylindrical coordinate system (x, r, θ) with velocity components u_x, u_r, and u_θ [the x axis is directed along the wedge apex (Fig. 1)] have the form

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} = 0; \tag{1.1}$$

$$\frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x} + u_r\frac{\partial u_x}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_x}{\partial \theta} + \frac{\partial p}{\partial x} = 0; \tag{1.2}$$

$$\frac{\partial u_r}{\partial t} + u_x\frac{\partial u_r}{\partial x} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} + \frac{\partial p}{\partial r} = 0; \tag{1.3}$$

$$\frac{\partial u_\theta}{\partial t} + u_x\frac{\partial u_\theta}{\partial x} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} + \frac{1}{r}\frac{\partial p}{\partial \theta} = 0. \tag{1.4}$$

The moving boundaries of the wedge $\theta = \theta_1(t)$ and $\theta = \theta_2(t)$ are subjected to the no-slip conditions:

$$u_\theta(r, \theta_{1,2}, t) = r\theta'_{1,2}. \tag{1.5}$$

The transport equation for the axial component of vorticity $\omega = (1/r)\partial\,(ru_\theta)/\partial r - (1/r)\partial u_r/\partial\theta$ is

$$\frac{\partial \omega}{\partial t} + u_x\frac{\partial \omega}{\partial x} + u_r\frac{\partial \omega}{\partial r} + \frac{u_\theta}{r}\frac{\partial \omega}{\partial \theta} = \omega_x\frac{\partial u_x}{\partial x} + \omega_r\frac{\partial u_x}{\partial r} + \frac{\omega_\theta}{r}\frac{\partial u_x}{\partial \theta}. \tag{1.6}$$

In the case of a planar flow, the stream function ψ is introduced, and condition (1.5) is integrated:

$$\psi(r, \theta_{1,2}, t) = -r^2\theta'_{1,2}/2. \tag{1.7}$$

Here $ru_2 = \partial\psi/\partial\theta$ and $u_\theta = -\partial\psi/\partial r$.

If the planar flow is steady, then the vorticity is a prescribed function of ψ: $\omega = \omega(\psi)$. The equation for determining the function ψ has the form

$$\frac{1}{r}\frac{\partial}{\partial r}\Big(r\,\frac{\partial\psi}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = -\omega. \tag{1.8}$$

Condition (1.7) can be simplified as

$$\psi(x, r, 0) = \psi(x, r, \theta_0) = 0. \tag{1.9}$$

Here $\theta = 0$ and $\theta = \theta_0$ are the wedge faces.

2. Planar Steady Flow. Based on the theorem of expansion (see [9]), the planar and spatial flows can be considered as a superposition of potential and vortex flows. Dividing the motion inside the wedge into the potential component ψ_1 and vorticity component ψ_2, we can present the stream function as their superposition:

$$\psi(r, \theta) = \psi_1(r, \theta) + \psi_2(r, \theta), \qquad \Delta\psi_1 = 0, \qquad \Delta\psi_2 = -\omega. \tag{2.1}$$

If the components have different orders of smallness as $r \to 0$, the no-slip condition (1.9) is imposed for each component separately:

$$\psi_1(r, 0) = \psi_1(r, \theta_0) = 0, \qquad \psi_2(r, 0) = \psi_2(r, \theta_0) = 0.$$

If the potential component prevails, the vorticity is transported in a steady flow as an impurity, following the vortex-free flow streamlines, because $\omega = \omega(\psi_1 + \psi_2) \approx \omega(\psi_1)$.

We expand the function ψ_1 into the Fourier series, where we need the first term only:

$$\psi_1 = c_1 r^{\pi/\theta_0} \sin(\pi\theta/\theta_0) + o(r^{\pi/\theta_0}). \tag{2.2}$$

There is no singularity for velocity at the wedge apex if $\theta_0 \leqslant \pi$. By analogy with the theory of axisymmetric conical flows [4, 5], the dependence of ω on ψ can be presented in a power form $\omega(\psi) = a(\psi_1 + \psi_2)^k$, where a is a constant. There is no singularity for the vorticity on the body if $k \geqslant 0$. From Eq. (1.8), we obtain

$$\frac{1}{r}\frac{\partial}{\partial r}\Big(r\,\frac{\partial\psi_2}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2\psi_2}{\partial\theta^2} = -a\Big(c_1 r^{\pi/\theta_0}\sin\frac{\pi\theta}{\theta_0} + \psi_2\Big)^k. \tag{2.3}$$

The solution is presented in a power form as

$$\psi_2 = r^n f(\theta) + o(r^n). \tag{2.4}$$

As $r \to 0$, there is no singularity for the vorticity if $n \geqslant 2$. The solution depends on the relation between the parameters n and θ_0. If $n\theta_0 > \pi$, the vorticity is transported along the streamlines of a vortex-free flow. From Eq. (2.3), we obtain $f'' + n^2 f = b\sin^k(\pi\theta/\theta_0)$, where $b = -ac_1^k$ and $n = 2 + k\pi/\theta_0$. The solution has the form

$$f = A(\theta)\sin[n(\theta - \chi)],$$

$$A = b\int_0^\theta \sin^{-2n}[n(\theta - \chi)]\,d\theta \int_0^\theta \sin^{2n}[n(\theta - \chi)]\sin^k\Big(\frac{\pi\theta}{\theta_0}\Big)\,d\theta,$$

where χ is a constant. The boundary condition at $\theta = \theta_0$ can be satisfied by two methods, adopting the condition $A(\theta_0) = 0$ or $\sin[n(\theta - \chi)] = 0$.

If $n\theta_0 = \pi$, superposition (2.1) is unreasonable, and the problems of potential and vortex flows should be considered jointly. Then, this case is identical to the case with $n\theta_0 < \pi$, and Eq. (2.3) yields

$$f'' + n^2 f + af^k = 0, \qquad kn = n - 2, \tag{2.5}$$

where $\operatorname{Im} f^k = 0$.

After integrating Eq. (2.5), we obtain

$$f' = \pm A(f), \tag{2.6}$$

where $A = C^2 - n^2 f^2 - 2af^{k+1}/(k + 1)$.

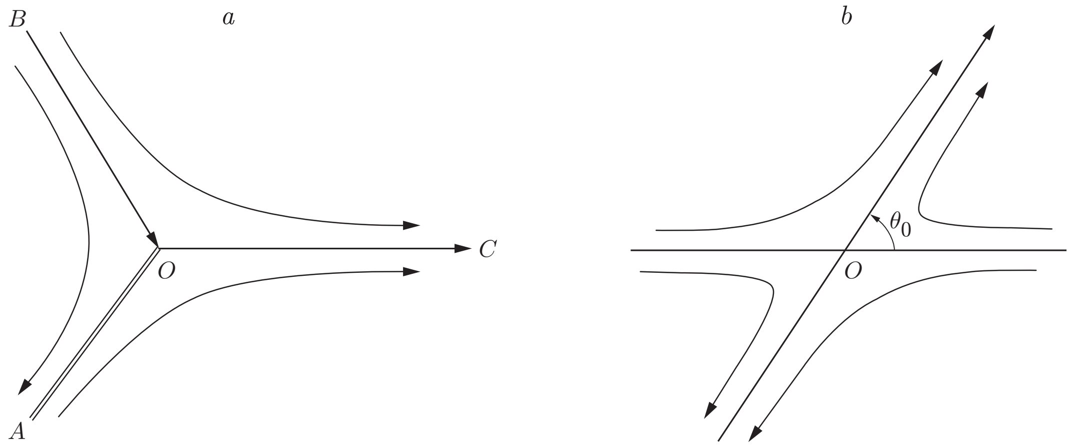

Fig. 2. Extension of the solution to the entire plane: (a) potential flow with $N = 3$; (b) vortex flow ($N = 4$ and $\theta_0 = \pi/2$).

Without loss of generality, we consider the case with $C = f'(0) > 0$. The segment $0 \leqslant \theta \leqslant \theta_0$ should have at least one maximum $f_{\max}$, because $f = 0$ at the ends of this segment. Therefore, the condition of existence of the solution of Eq. (2.6) is the equality $A = 0$ at a certain (at least one) value of f_0. As the solution is symmetric with respect to this maximum, it is reached at $\theta = \theta_0/2$:

$$f_{\max} = f(\theta_0/2), \qquad C^2 = n^2 f_{\max}^2 + 2a f_{\max}^{k+1}/(k+1).$$

Then, $f' = A$ if $0 \leqslant \theta \leqslant \theta_0/2$ and $f' = -A$ if $0 \leqslant \theta \leqslant \theta_0/2$. The problem does not have a solution for all values of the parameters a and k. Thus, if $k = 1$ and $a > -n^2$, then $A \neq 0$, i.e., there is no solution. If $k \neq 1$, then the condition of existence of the solution can be obtained for small values of θ_0 ($\theta_0 \ll 1$). Then we obtain $dF/dx \approx (C^2 - 2aF^{k+1}/(k+1))^{1/2}$, where $f = \theta_0^{2/(1-k)} F$ and $\theta = \theta_0$. The solution exists if $2a/(k+1) > 0$.

Particular solutions of Eq. (2.6) are also found in two cases where the problem is linearized.

1. For $n = 2$, the solution is the sum of two harmonics: $f = (a/4)(\tan\theta_0 \sin 2\theta + \cos 2\theta - 1)$.

2. As $n \to \infty$ and $k \to 1$, the effect of vorticity in Eq. (2.5) is significant if $a \to a_1 n^2$ and $a_1 = O(1)$. Assuming that $\vartheta = n\theta$, $\vartheta_0 = n\theta_0 = \pi(1 + a_1)^{-1/2}$, and $f = f(\vartheta)$, we obtain $f'' + (1 + a_1) f = 0$. The solution of this equation $f = c_1 \sin[\vartheta(1 + a_1)^{-1/2}]$ exists if $a_1 > -1$.

The solution determined in the interval $0 \leqslant \theta \leqslant \theta_0$ may be symmetrically extended to the entire plane $0 \leqslant \theta \leqslant \pi$ by separating the latter into sectors (cells) whose sides form convergence lines. As the power exponent in solution (2.2) depends on the sector angle θ_0, the Bernoulli law predicts that the pressure on the limiting streamlines is identical on different sides of the sector only if the angles of the sectors are identical and equal to $2\pi/N$ (N is an integer). The convergence line may be simultaneously the line of the tangential discontinuity in velocity. The potential flow for $N = 3$ is schematically shown in Fig. 2a (AO is the line of the tangential discontinuity in velocity). A flow pattern with three discontinuities (AO, BO, and CO) is also possible.

As the exponent n in the general case is independent of θ_0, solution (2.5) admits conjugation in various sectors. Figure 2b shows the topology of the streamlines of solution (2.6) extended to the entire plane; the flow is continuous. In each of the four sectors, the flow direction indicated by the arrows can be changed to the opposite, which yields a set of discontinuous solutions. It should be noted that it is always possible to introduce a velocity discontinuity on an arbitrarily chosen unbounded surface in a steady inviscid flow by changing the direction of the flow below (above) one of the sides of this surface to the opposite flow direction.

Figure 3 shows the possible variants of the discontinuous flow around the body, which were obtained from continuous flows.

3. Planar Unsteady Flow. The vorticity distribution being known, Eq. (1.8) is valid for both steady and unsteady problems. If the no-slip condition on the wedge faces is satisfied and the wedge faces are motionless, the difference between the problems implies that the constants entering the local steady solution become arbitrary functions of time in the unsteady case, which have to be found from the solution of the global problem. Such a flow, with unsteadiness generated far from the wedge apex, will be called a quasi-steady flow.

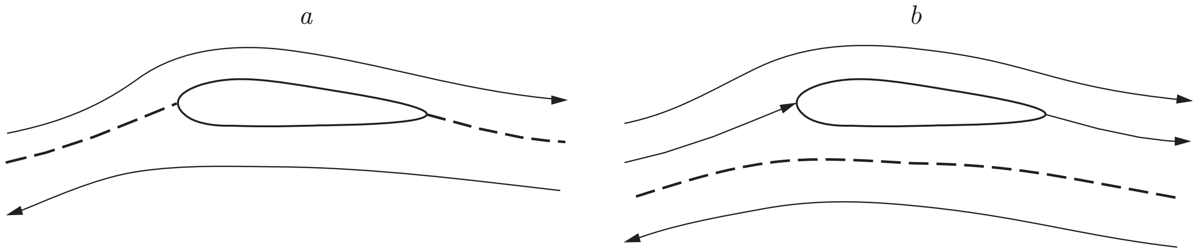

Fig. 3. Patterns of discontinuous flows around the body: (a) discontinuity coinciding with the zero streamline adjacent to the body; (b) discontinuity coinciding with the zero streamline not adjacent to the body; the dashed curve shows the contact discontinuity.

A principal difference between quasi-steady and unsteady problems allows us to find an example where the wedge faces are rotated around the apex by a prescribed law $\theta_{1,2} = \theta_{1,2}(t)$. As the rotation of the faces does not generate the vorticity, the latter is formed far from the apex. The rotation of the faces affects only the vorticity transport.

The motion inside the wedge includes three components: potential steady, potential unsteady, and vorticity components. Hence, the stream function can be presented as a sum of three components:

$$\psi(r,\theta,t) = \psi_1(r,\theta) + \psi_2(r,\theta,t) + \psi_3(r,\theta,t).$$

The boundary conditions (1.7) acquire the form

$$\psi_1(r,\theta_{1,2}) = 0, \qquad \psi_2(r,\theta_{1,2},t) = 0, \qquad \psi_3(r,\theta_{1,2}) = -r^2\theta'_{1,2}/2.$$

The function ψ_1 can be presented in the form (2.2), and the function ψ_3 can be written in a closed form as

$$\psi_3 = r^2[\alpha(t)\sin 2\theta + \beta(t)\cos 2\theta].$$

Thereby, we have $\alpha\sin 2(\theta_2-\theta_1) = \theta'_1\cos 2\theta_2 - \theta'_2\cos 2\theta_1$ and $\beta\sin 2(\theta_2-\theta_1) = \theta'_2\sin 2\theta_1 - \theta'_{12}\sin 2\theta_2$. The solution is singular if $\theta_2-\theta_1 = \pi/2$, π, or $3\pi/2$. The contribution of the unsteady solution ψ_3 is greater than the contribution of the solution ψ_1 for $\theta_0 = \theta_2-\theta_1 < \pi/2$ and is small for $\theta_0 > \pi/2$.

As applied to a planar flow ($\partial/\partial x = 0$), Eq. (1.6) is simplified:

$$\frac{\partial\omega}{\partial t} + u_r\,\frac{\partial\omega}{\partial r} + \frac{u_\theta}{r}\,\frac{\partial\omega}{\partial\theta} = 0. \tag{3.1}$$

The function ψ_2 is sought in the form (2.4).

If the potential flow velocities are low, i.e., $m > 2$ and $n > 2$, the vorticity transport is insignificant, and the initial distribution of vorticity is not subjected to changes: $\partial\omega/\partial t = 0$. In accordance with Eq. (1.8), we have $\omega = O(r^{n-2})$.

Unsteadiness is significant for $n = m = 2$. Presenting the solution in the general form $\psi = r^2\Psi(\theta,t)$, $\omega = \omega(\theta,t)$, we obtain the following system of equations from Eqs. (1.8) and (3.1):

$$\frac{\partial\omega}{\partial t} - 2\Psi\,\frac{\partial\omega}{\partial\theta} = 0, \qquad \frac{\partial^2\Psi}{\partial\theta^2} + 4\Psi + \omega = 0. \tag{3.2}$$

System (3.2) admits a solution of the form $\Psi = \Phi(\theta)/t$, $\omega = \Omega(\theta)/t$, which describes vorticity amplification as $t \to -0$ or its decay as $t \to \infty$. Equations (3.2) become ordinary:

$$2\Phi\Omega' + \Omega = 0, \qquad \Phi'' + 4\Phi + \Omega = 0. \tag{3.3}$$

Let us again consider the case with a small apex angle of the wedge: $\theta_0 \ll 1$. Let $\theta = \theta_0\vartheta$, $\Phi = \theta_0 f(\vartheta)$, and $\Omega = \gamma(\vartheta)/\theta_0$. Then, system (3.3) becomes even more simplified: $2f\gamma' + \gamma = 0$ and $f'' + \gamma = 0$. This system is equivalent to one third-order equation for the function f: $2ff''' + f'' = 0$. Integrating this equation two times, we obtain

$$f = a_2(1 + a_1 - f')^{(1+a_1)/(1+2a_1)}(a_1 + f')^{a_1/(1+2a_1)},$$

where a_1 and a_2 are arbitrary positive constants, f' is an implicit two-valued function of f (on the wedge faces, $f = 1 + a_1$ and $f = -a_1$). The maximum value of f is reached at the line of flow turning ($u_r = 0$): $f = (1+a_1)^{(1+a_1)/(1+2a_1)}a_1^{a_1/(1+2a_1)}$.

4. Spatial Flows. An analysis of a three-dimensional flow is rather difficult; hence, we consider examples where the dependence on the third coordinate x either is known *a priori* or can be readily found from simple considerations.

4.1. *Flow Accelerated along a Dihedral Angle.* The exact solution of the Euler equations is the Burgers–Rott vortex "locked" in a dihedral angle [10, 11]:

$$u_x = ax, \qquad u_r = rv(\theta), \qquad u_\theta = rw(\theta), \qquad p = r^2 p_0(\theta) - a^2x^2/2$$

(a is a constant). Equations (1.1)–(1.4) yield the system

$$w\omega' - a\omega = 0, \qquad w'' + 4w - 2\omega = 0, \tag{4.1}$$

which coincides with the considered system (3.3) with accuracy to notation and a constant factor. For $\omega = 0$, the solution is trivial: $w = A\sin 2\theta$ and $\theta_0 = \pi/2$.

4.2. *Unsteady Analogy.* The equivalence between planar unsteady motion described by Eqs. (3.3) and three-dimensional steady motion described by Eqs. (4.1) allows us to assume that there is an analogy in the case of a constant (in the first approximation) streamwise component of velocity (as in the theory of a slender body). Let

$$u_x = u_\infty + o(1), \qquad p = r^2 p_0(\theta, x) + o(r^2),$$

$$u_r = rv(\theta, x) + o(r), \qquad u_\theta = rw(\theta, x) + o(r).$$

From the equations of continuity (1.1) and vorticity transport (1.6), we obtain

$$u_\infty \frac{\partial \omega}{\partial x} + w\frac{\partial \omega}{\partial \theta} = 0, \qquad \frac{\partial \omega}{\partial \theta} + 2v = 0, \tag{4.2}$$

where $\omega = 2w - \partial v/\partial\theta$.

The coordinate $x = u_\infty t$ plays the role of time. The equation for the streamwise velocity is solved independent of system (4.2). The vorticity is transported in the x direction along a certain spiral.

Let us consider the structure of singularity in a certain cross section $x = 0$, assuming that $x\omega = u_\infty a(\theta)$, $xw = u_\infty b(\theta)$, and $xv = u_\infty c(\theta)$. Eliminating $c(\theta)$ from Eq. (4.2), we obtain

$$b'' + 8b - 4a = 0, \qquad ba' - a = 0. \tag{4.3}$$

Equation (4.3) is again reduced to system (3.3) by mere stretching of variables.

4.3. *Linear Wave.* We consider a harmonic wave "locked" in a dihedral angle:

$$u_x = r^{n-1}\exp{(ikt)}u(x,\theta) + o(r^{n-1}), \qquad u_r = r^n \exp{(ikt)}v(x,\theta) + o(r^n),$$

$$u_\theta = r^n \exp{(ikt)}w(x,\theta) + o(r^n), \qquad p = r^{n-1}\exp{(ikt)}p_0(x,\theta) + o(r^{n-1}).$$

The nonlinear terms in (1.1)–(1.4) are small; hence, we have

$$iku + \frac{\partial p_0}{\partial x} = 0, \qquad ikv + (n-1)p_0 = 0, \qquad ikw + \frac{\partial p_0}{\partial \theta} = 0, \qquad \frac{\partial u}{\partial x} + (n+1)v + \frac{\partial w}{\partial \theta} = 0.$$

The flow is vortex-free. From Eq. (3.2), we derive the equation for determining w:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial \theta^2} + (n^2 - 1)w = 0.$$

The solution $w = w_0(x)\sin{(\pi\theta/\theta_0)}$ describes a standing wave for $(\pi/\theta_0)^2 < n^2 - 1$ and an exponential flow for $(\pi/\theta_0)^2 > n^2 - 1$.

5. Planar Analog of a Quasi-Conical Flow. A steady quasi-conical flow is formed in regions bounded by tangent curves, which are described by the power law $y = x^n\xi_{1,2}$ ($\xi_{1,2}$ are constants). Examples of quasi-conical flows are shown in Fig. 4. In contrast to the conical solution, the quasi-conical solution is approximate and is valid as $x \to +0$ in the interval $\xi_2 \leqslant \xi \leqslant \xi_1$. We consider the case $n > 1$ (Fig. 4a) assuming, similar to (2.1), that $\psi(x,\xi) = \psi_1(x,\xi) + \psi_2(x,\xi)$, $\Delta\psi_1 = 0$, $\Delta\psi_2 = -\omega(\psi)$, $\omega = a\psi^k$, $y = \xi x^n$, and $k \geqslant 0$. Such a flow is formed in the region behind the laminar separation. The solution of the problem is obtained by the method of separation of variables with allowance for the boundary conditions: $\psi_1(x,\xi_{1,2}) = \psi_2(x,\xi_{1,2}) = 0$. Assuming that $\psi_1 = A(x)\varphi(\xi)$, we obtain $x^{2n}A''\varphi + A\varphi'' = 0$ from the Laplace equation in the first approximation. The solution has the form

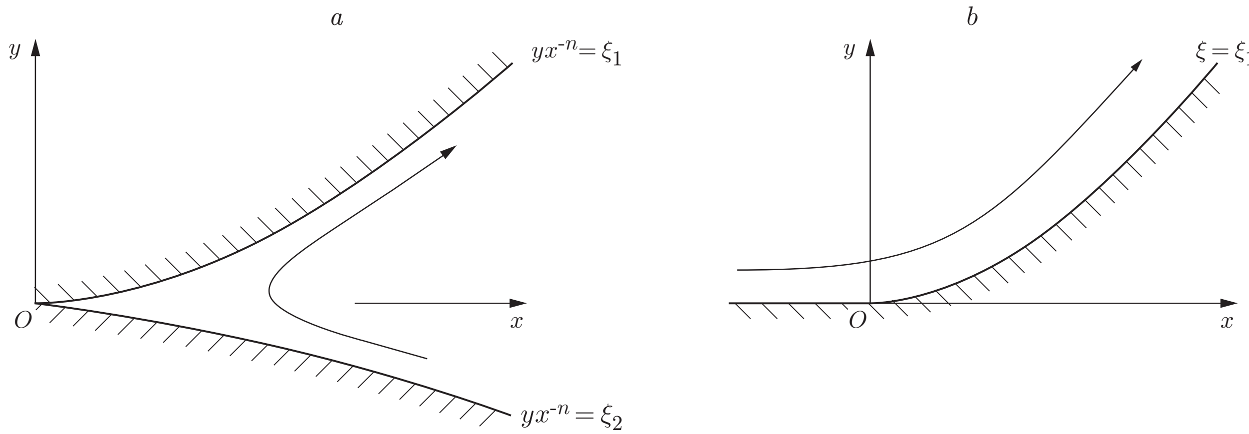

Fig. 4. Patterns of quasi-conical flows inside (a) and outside (b) the “zero” angle.

$$\varphi = \sin[s(\xi - \xi_2)], \qquad A = \exp[-sx^{1-n}/(n-1)],$$

where $s = \pi/(\xi_1 - \xi_2) > 0$.

It follows from the method of the solution that the function Φ is determined with accuracy to an arbitrary factor, which is a function $B(x)$ satisfying the condition $|B'| = o(x^{-n})$.

The solution for ψ_2 is also sought by the method of separation of variables:

$$\psi_2 = x^m f(\xi) + o(x^m). \tag{5.1}$$

As the potential flow is exponentially small, we have $\omega = a(\psi_1 + \psi_2)^k \approx a\psi_2^k$. The equation for determining ψ_2 is independent of ψ_1

$$x^{m-2n} f'' \approx -a\psi_2^k = -ax^{km} f^k$$

and reduces to the ordinary differential equation

$$f'' + af^k = 0, \tag{5.2}$$

where $k = 1 - 2n/m$ and $0 < k \leqslant 1$.

The solution of Eq. (5.2) is obtained in an implicit form

$$(\xi_1 - \xi_2) \int\limits_0^z \frac{dz}{\sqrt{1 - z^{1+k}}} = (\xi - \xi_2) \int\limits_0^1 \frac{dz}{\sqrt{1 - z^{1+k}}},$$

where $z = f/f_{\max}$, $0 \leqslant z \leqslant 1$, and $f_{\max} = f((\xi_1 + \xi_2)/2)$.

The problem of an external flow around a “zero” angle (Fig. 4b) describes the flow in a pre-separation region. The solution of the vortex-free problem is trivial: $\psi_1 = u_\infty x^n(\xi - \xi_1) + o(x^n)$. For $m = n$, the solution of the vortex problem is unphysical, because there is a singularity in the vorticity distribution ($k = -1$). For $m > n$, we have a power-type solution in the form (5.1):

$$f = c(\xi - \xi_1) + \frac{b}{(k+1)(k+2)}\,(\xi - \xi_1)^{k+2}$$

(c is a constant; $k = m/n - 2$).

Conclusions. The theory of conical flows developed for supersonic velocities is also applicable to an incompressible fluid flow. The local ($r \to 0$) solution constructed in the form of a coordinate series may be the exact solution of the Euler equation if this series is terminated: its terms are rejected beginning from a certain term (normally, the second one). Such examples are known for both compressible and incompressible conical flows.

The theory of conical flows describes the singularity at the point $r = 0$. This fact is important in the numerical solution of the global problem, because either the calculation can be started from the point $r = 0$ or the neighborhood of this point can be skipped in calculations as being known in advance.

The analysis of conical and quasi-conical flows is difficult because the problem contains an arbitrary function: the initial distribution of the velocity rotor. It turned out that the solution of the local problem does not exist for all distributions of the velocity rotor. The question about the relation between the absence of solutions of the local and global problems remains open. The local theory is invalid at large distances, i.e., as $r \to \infty$.

REFERENCES

1. U. Frisch, *Turbulence. The Legacy of A. N. Kolmogorov*, Cambridge University Press, Cambridge (1995).
2. R. Temam, *Navier–Stokes Equations. Theory and Numerical Analysis*, North-Holland, Amsterdam (1977).
3. J. Fernandez-de la Mora, R. Fernandes-Feria, M. Peres-Saborid, and A. Barrero, "Conically similar swirling flows at high Reynolds numbers," *Quart. J. Mech. Appl. Math.*, **52**, Part 1, 1–53 (1999).
4. F. I. Frankl and E. A. Karpovich, *Gas-Dynamics of Slender Bodies* [in Russian], Gostekhteoretizdat, Moscow–Leningrad (1948).
5. S. K. Betyaev, "Conical and quasi-conical flows of an incompressible fluid," *Izv. Ross. Akad. Nauk, Mekh. Zhidk. Gaza*, No. 4, 89–100 (2003).
6. S. K. Betyaev and A. M. Gaifullin, *Helical Vortices* [in Russian], Central Aerohydrodynamic Inst., Moscow (2001).
7. M. Van-Dyke, *Perturbation Methods in Fluid Mechanics*, Academic Press, New York–London (1964).
8. G. I. Barenblatt, *Scaling, Self-Similarity, and Intermediate Asymptotics*, Cambridge University Press, Cambridge (1996).
9. M. E. Goldshtein, *Aeroacoustics* [in Russian], Mashinostroenie, Moscow (1981).
10. P. G. Saffman, *Vortex Dynamics*, Cambridge University Press, Cambridge (1992).
11. S. V. Alexeenko, P. A. Kuibin, and V. L. Okulov, *Introduction into the Theory of Concentrated Vortices* [in Russian], Inst. Computer Research, Izhevsk, Moscow (2002).